Modern approaches to quantum gravity

Homework 4 Fall 2025

1. Large-N gauge theory

Consider the $U(N)$ Yang-Mills theory with a gauge field $A_\mu = A_\mu^a T^a$ in the adjoint representation. The field strenth is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \tag{1}$$

Let the functional integral of the theory with a Yang-Mills coupling parametrised in terms of the 't Hooft coupling $\lambda \equiv N g_{YM}^2$

$$Z[\lambda] = \int \mathcal{D}A \ \exp\left\{-\frac{N}{\lambda}\int d^4x \ \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})\right\} \tag{2}$$

(a) Given a local single trace operator $\mathcal{O}(x)$ (e.g. $\mathrm{Tr}(F_{\mu\nu}^5(x))$), consider the corresponding generating functional

$$Z[J] = \int \mathcal{D}A \ \exp\left\{-\frac{N}{\lambda}\int d^4x \ \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) - N\int d^4x \ \mathcal{O}(x)J(x)\right\} \tag{3}$$

Express the n-point correlation function of $\mathcal{O}$, $\langle \mathcal{O}(x_1)...\mathcal{O}(x_n)\rangle$ as a functional derivative of the generating functional in (3).
Since $\mathcal{O}(x)$ is a gauge-invariant function of the fields, the generating functional can again be expressed as a sum of diagrams.
Argue that there exist functionals $F_0, F_1...$ such that (3) can be written as

$$Z[J] = \exp\left\{N^2 F_0[J,\lambda] + N^0 F_1[J,\lambda] + \mathcal{O}(N^{-2})\right\} \tag{4}$$

where $F_0[J,\lambda]$ contains the contributions of planar bubble diagrams, and subleading terms contain the contributions of higher genus diagrams.
Note that this means that large-N gauge theories behave semiclassically, with an effective Planck's constant $h_{eff} = \frac{h}{N^2}$.
Hint: Use the fact that the scaling of connected bubble diagrams is $\mathrm{diag}(V,E,F) \sim N^\chi$, regardless of how many edges meet at every interaction vertex.

(b) Compute the two point function $\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle$ and identify the leading connected and disconnected contributions. Show that the disconnected contributions dominate in the large-N limit, i.e.

$$\frac{\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle_{conn}}{\langle \mathcal{O}(x_1)\rangle\langle \mathcal{O}(x_2)\rangle} \sim N^{-2} \tag{5}$$

(c) Let $\mathcal{O}_i, \ i = 1,...n$ be single trace operators in the gauge theory, with $n \ll N$. Show that the correlation functions are dominated by the disconnected contributions:

$$\left\langle \prod_i \mathcal{O}_i \right\rangle = \prod_i \langle \mathcal{O}_i\rangle(1 + \mathcal{O}(N^{-2})) \tag{6}$$

where we suppressed the dependence on x_i for brevity.
Why is the assumption $n \ll N$ necessary?
Hint: the generating functional now takes the more general form

$$Z[J_i] = \exp\{N^2 F_0[J_i] + \mathcal{O}(N^0)\} \tag{7}$$

(d) Let us now consider mean-subtracted operators

$$\tilde{\mathcal{O}}_i \equiv \mathcal{O}_i - \langle \mathcal{O}_i \rangle \tag{8}$$

Prove that correlation functions of these operators now factorise into connected 2-point functions according to the Wick-like contraction procedure

$$\left\langle \prod_i \tilde{\mathcal{O}}_i \right\rangle = \prod_{\substack{i<j \\ \text{Wick}}} \langle \tilde{\mathcal{O}}_i \tilde{\mathcal{O}}_j \rangle (1 + \mathcal{O}(N^{-2})) \tag{9}$$

where $\prod_{\substack{i<j \\ \text{Wick}}}$ denotes a product over all combinations of i, j where no index is repeated (i.e. every operator appears in exactly one 2-point correlation function). Operators that satisfy these type of relations are called *Generalised Free Fields.*

2. Polyakov action and stress-tensor modes

(a) Consider the 'Polyakov' action for a massive point particle moving in Minkowski space

$$S[X, e] = \int \left(e^{-1}(\tau) \eta_{\mu\nu} \frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\tau} - e(\tau) m^2 \right) d\tau \tag{10}$$

Show that, after substituting the equation of motion for $e(\tau)$, one recovers the action of a relativistic point particle.

(b) Higher-dimensional objects are called *branes.* More precisely, if the objects has p spatial dimensions, it's called a p-brane. The dynamics of a p-brane moving in Minkowski space is given by the Dirac action,

$$S = -T \int d^{p+1}\sigma \sqrt{-\det \gamma} \tag{11}$$

Here σ^α are coordinates on the brane worldvolume, with $\alpha = 0, ..., p$, while

$$\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu} \tag{12}$$

Show that this is equivalent to the Polyakov-type action with dynamical worldvolume metric $g_{\alpha\beta}$

$$S = -\frac{T}{2} \int d^{p+1}\sigma \sqrt{-g} (g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - (p-1)) \tag{13}$$

(c) Let us now turn to the case of a closed string ($p = 1$). In worldsheet light-cone coordinates $\sigma^{\pm} = \tau \pm \sigma$, one can write the energy momentum tensor components as

$$T_{++} = -\frac{1}{\alpha'}\eta_{\mu\nu}\partial_+ X^\mu \partial_+ X^\nu , \quad T_{--} = -\frac{1}{\alpha'}\eta_{\mu\nu}\partial_- X^\mu \partial_- X^\nu , \quad T_{+-} = T_{-+} = 0 \tag{14}$$

Writing the mode expansion of the target space coordinates X^μ as

$$X^\mu = x^\mu + \frac{\alpha'}{2} p^\mu (\sigma^- + \sigma^+) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \left[\alpha_n^\mu e^{-in\sigma^-} + \tilde{\alpha}_n^\mu e^{-in\sigma^+} \right], \tag{15}$$

find an expression for the modes ℓ_n, $\tilde{\ell}_n$ which are defined in light-cone coordinates by

$$T_{--}(\sigma^-) = -\sum_n \ell_n e^{-in\sigma^-} \quad T_{++}(\sigma^+) = -\sum_n \tilde{l}_n e^{-in\sigma^+} \tag{16}$$

Note that by inverting the Fourier series one can express

$$\ell_n = -\frac{1}{2\pi} \int_0^{2\pi} d\sigma T_{--}(\sigma) e^{in\sigma} , \qquad \tilde{\ell}_n = -\frac{1}{2\pi} \int_0^{2\pi} d\sigma T_{++}(\sigma) e^{in\sigma} \tag{17}$$

(d) Using the Poisson bracket relation $\{\alpha_m^\mu, \alpha_n^\nu\} = -im\eta^{\mu\nu}\delta_{m+n,0}$, show that $\{\ell_m, \alpha_n^\mu\} = in\alpha_{m+n}^\mu$ and hence that

$$\{\ell_m, \ell_n\} = -i(m-n)\ell_{m+n} \tag{18}$$

This is the so-called Witt algebra, whose centrally-extended version is the Virasoro algebra.